Exercise 40

If $x^2 + xy + y^3 = 1$, find the value of y''' at the point where $x = 1$.

Solution

Differentiate both sides with respect to x.

$$\frac{d}{dx}(x^2 + xy + y^3) = \frac{d}{dx}(1)$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(xy) + \frac{d}{dx}(y^3) = 0$$

$$(2x) + \left[\frac{d}{dx}(x)\right] y + x\left[\frac{d}{dx}(y)\right] + \left[(3y^2)\cdot\frac{d}{dx}(y)\right] = 0$$

$$2x + (1)y + x(y') + (3y^2)(y') = 0$$

$$2x + y + xy' + 3y^2y' = 0$$

Solve for y'.

$$2x + y + (x + 3y^2)y' = 0 \tag{1}$$

$$y' = -\frac{2x + y}{x + 3y^2}$$

Differentiate both sides of equation (1) with respect to x to get y''.

$$\frac{d}{dx}[2x + y + (x + 3y^2)y'] = \frac{d}{dx}(0)$$

$$\frac{d}{dx}(2x) + \frac{d}{dx}(y) + \frac{d}{dx}[(x + 3y^2)y'] = 0$$

$$(2) + (y') + \left[\frac{d}{dx}(x + 3y^2)\right] y' + (x + 3y^2)\frac{d}{dx}(y') = 0$$

$$2 + y' + (1 + 6y\cdot y')y' + (x + 3y^2)y'' = 0$$

Bring the terms with y' to the right side.

$$\begin{aligned}
(x + 3y^2)y'' &= -2 - 2y' - 6y(y')^2 \\
&= -2 - 2\left(-\frac{2x + y}{x + 3y^2}\right) - 6y\left(-\frac{2x + y}{x + 3y^2}\right)^2 \\
&= -\frac{2(x + 3y^2)^2}{(x + 3y^2)^2} + \frac{2(2x + y)(x + 3y^2)}{(x + 3y^2)^2} - \frac{6y(2x + y)^2}{(x + 3y^2)^2} \\
&= \frac{-2(x + 3y^2)^2 + 2(2x + y)(x + 3y^2) - 6y(2x + y)^2}{(x + 3y^2)^2}
\end{aligned}$$

Expand the numerator and solve for y''.

$$(x+3y^2)y'' = \frac{2x^2 + 2xy - 24x^2y - 24xy^2 - 18y^4}{(x+3y^2)^2}$$

$$y'' = \frac{2x^2 + 2xy - 24x^2y - 24xy^2 - 18y^4}{(x+3y^2)^3}$$

Multiply both sides by $(x+3y^2)^3$.

$$(x+3y^2)^3 y'' = 2x^2 + 2xy - 24x^2y - 24xy^2 - 18y^4$$

Differentiate both sides with respect to x once more to get y'''.

$$\frac{d}{dx}[(x+3y^2)^3 y''] = \frac{d}{dx}(2x^2 + 2xy - 24x^2y - 24xy^2 - 18y^4)$$

$$\left[\frac{d}{dx}(x+3y^2)^3\right] y'' + (x+3y^2)^3 \left[\frac{d}{dx}(y'')\right] = \frac{d}{dx}(2x^2) + 2\frac{d}{dx}(xy) - 24\frac{d}{dx}(x^2y) - 24\frac{d}{dx}(xy^2) - 18\frac{d}{dx}(y^4)$$

$$3(x+3y^2)^2 \cdot (1+6yy')y'' + (x+3y^2)^3 y''' = (4x) + 2(y+xy') - 24(2xy + x^2y') - 24(y^2 + 2xyy') - 18(4y^3y')$$

Plug in $x = 1$ to the given equation to find the corresponding y-value on the curve.

$$x = 1: \quad (1)^2 + (1)y + y^3 = 1 \quad \to \quad y = 0$$

So plug in $x = 1$ and $y = 0$ into the formula involving y'''.

$$3 \cdot (1)y''(1,0) + (1)^3 y'''(1,0) = (4) + 2[y'(1,0)] - 24[y'(1,0)] - 24(0) - 18(0)$$

$$3y''(1,0) + y'''(1,0) = 4 - 22y'(1,0)$$

Solve for $y'''(1,0)$ and evaluate the lower derivatives at $x = 1$ and $y = 0$.

$$\begin{aligned} y'''(1,0) &= 4 - 22y'(1,0) - 3y''(1,0) \\ &= 4 - 22\left[-\frac{2(1)+0}{1+3(0)^2}\right] - 3\left\{\frac{2(1)^2 + 2(1)(0) - 24(1)^2(0) - 24(1)(0)^2 - 18(0)^4}{[1+3(0)^2]^3}\right\} \\ &= 42 \end{aligned}$$